ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Sound and Vibration 260 (2003) 807–827

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Free vibration study of layered cylindrical shells by collocation with splines

K.K. Viswanathan[a,*], P.V. Navaneethakrishnan[b]

[a] Department of Mathematics, Crescent Engineering College, Anna University, Seethakathi Estate, GST Road, Vandalur, Chennai 600 048, India

[b] School of Mathematics, Anna University, Chennai 600 025, India

Received 26 November 2001; accepted 15 May 2002

Abstract

The free vibration of circular cylindrical thin shells, made up of uniform layers of isotropic or specially orthotropic materials, is studied using point collocation method and employing spline function approximations. The equations of motion for the shell are derived by extending Love's first approximation theory. Assuming the solution in a separable form a system of coupled differential equations, in the longitudinal, circumferential and transverse displacement functions, is obtained. These functions are approximated by Bickley-type splines of suitable orders. The process of point collocation with suitable boundary conditions results in a generalized eigenvalue problem from which the values of a frequency parameter and the corresponding mode shapes of vibration, for specified values of the other parameters, are obtained. Two types of boundary conditions and four types of layers are considered. The effect of neglecting the coupling between the flexural and extensional displacements is analysed. The influences of the relative layer thickness, a length parameter and a total thickness parameter on the frequencies are studied. Both axisymmetric and asymmetric vibrations are investigated. The effect of the circumferential node number on the vibrational behaviour of the shell is also analysed.

1. Introduction

Circular cylindrical shells are widely used in fields like aviation, rocketry, missiles and chemical and other industries. Layered composites are increasingly used in them because of their possible higher specific stiffness and better damping and shock absorbing characteristics over the homogeneous ones. In relation to the studies made and reported on vibrational behaviour of

*Corresponding author. Tel.: +91-44-237-6035; fax: +91-44-237-5620.
E-mail address: viswa20@hotmail.com (K.K. Viswanathan).

0022-460X/03/$ - see front matter © 2002 Published by Elsevier Science Ltd.
PII: S0022-460X(02)00923-9

circular cylindrical shells with homogeneous walls, the studies on such shells with laminated wall structure are only a few [1]. Baker and Herrmann [2] analysed three layered (Sandwich) shells. The theory developed was of 10th order including the effects of shear deformation, rotary inertia and initial stress. Markus [3] considered damped axisymmetric vibrations of two-layered cylindrical shells. The acoustic characteristics of such shells were studied by Chonan [4] assuming thick shell theory.

Dong [5] and Jones [6] used Donnell's [7] shallow shell theory while Bert et al. [8], Stavsky and Loewy [9] and Greenberg and Stavsky [10] used Love's [11] first approximation theory for their vibrational analyses of layered cylindrical shells. Significant work has been done of finite element modelling of layered anisotropic composite plates and shells [12]. Narita et al. [13] studied free vibration of angle-ply laminated cylindrical shell, using Ritz method and Flügge-type shell theory. Using a modification of Sander's shell theory and the third order shear deformation shell theory of Reddy, Khdeir and Reddy [14] analysed the dynamic and static behaviour of cross-ply laminated cylindrical and spherical incomplete shells.

The spline strip method was used by Mizusawa and Kito [15] to study the vibration of cross-ply laminated cylindrical panels. The method involved expressing displacement functions in a strip element as the product of basic function series in the axial direction and B-spline functions in the circumferential direction. Lam and Loy [16] considered the vibration of multi-layered cylindrical shell under nine different boundary conditions using Love's first approximation theory, with beam functions as axial modal functions, and the energy method. However, there seems to be no work carried so far on vibrational analysis of laminated cylindrical shells using Bickley spline function as done in the current analysis.

In this paper, the free vibrations of circular cylindrical shells made up of layers of constant thickness are analysed. The problem is formulated by extending Love's first approximation theory on homogeneous shells to these cases. The layers of the material are considered to be thin, linearly elastic and specially orthotropic or isotropic and assumed to be bonded perfectly together and to move without interface slip. The governing differential equations of motion are obtained in terms of the reference surface displacement components.

Four types of layer materials are considered in all and two sets of boundary conditions are imposed. Both axisymmetric and asymmetric vibrations are treated.

The differential equations of motion obtained are coupled in the longitudinal, circumferential and transverse displacement components. Assuming the solution in a separable form one gets a system of ordinary, but still coupled, differential equations on a set of assumed displacement functions which are functions of the meridional co-ordinate only. The equations have no closed-form solution in general. Numerical solution techniques have to be resorted to. In preference to a number of numerical methods available for such problems, like those of Galerkin, Runge–Kutta, Frobenius and Chebyshev collocation, a spline approximation technique is used. The basis for the choice of this method is that a chain of lower order approximations, as used here, can yield greater accuracy than a global higher order approximation. This conjecture was made and tested successfully by Bickley [17] over a two-point boundary value problem with a cubic spline. Subsequently, Soni and Sankara Rao [18], Irie et al. [19], Irie and Yamada [20], Navaneethakrishnan and Chandrasekaran [21], Navaneethakrishnan et al. [22], Navaneethakrishnan [23] and a few others have also demonstrated this, but most of them used only a single spline function in a problem.

In this work up to three displacement functions are approximated by splines, which are cubic or quintic, in a system of coupled equations. Their excellent convergence characteristics are brought out. These splines are simple and clear for analytical process and have significant computational advantage.

The solution procedure consists in modifying the differential equations obtained so as to admit quintic and cubic spline approximations for the displacement functions. Collocation with these splines yield a set of field equations which, along with the equations of boundary conditions, reduce to a system of homogeneous simultaneous algebraic equations on the assumed spline coefficients. The resulting generalized eigenvalue problem is solved for a frequency parameter using eigensolution technique to obtain as many eigenfrequencies as required, starting from the least. A search technique is also used at times. From the eigenvectors, the spline coefficients are computed from which the mode shapes are constructed.

Since the effect of coupling between bending and stretching is most significant when the number of layers is two, only two-layered shells are considered for detailed study.

Extensive parametric studies are made. The effects of a length parameter, the relative thickness of the layers, the overall thickness-to-radius ratio, the circumferential wave number and the boundary conditions on the frequency parameter are analysed. The effect of neglecting the coupling between bending and stretching is also investigated. Significant mode shapes are presented. The results are presented in terms of graphs and tables and discussed.

2. Formulation

A thin circular cylindrical shell made up of uniformly thick layers is considered. Each layer is assumed to be homogeneous, linearly elastic and isotropic or specially orthotropic, with the lines of orthotropy coinciding with the principal axes of the shell surface. The layers are perfectly bonded at the interfaces.

The geometry of the laminated cylindrical shell of constant thickness and the arrangement of its layers are clarified in Fig. 1. The thickness of the layers are not completely independent. Their

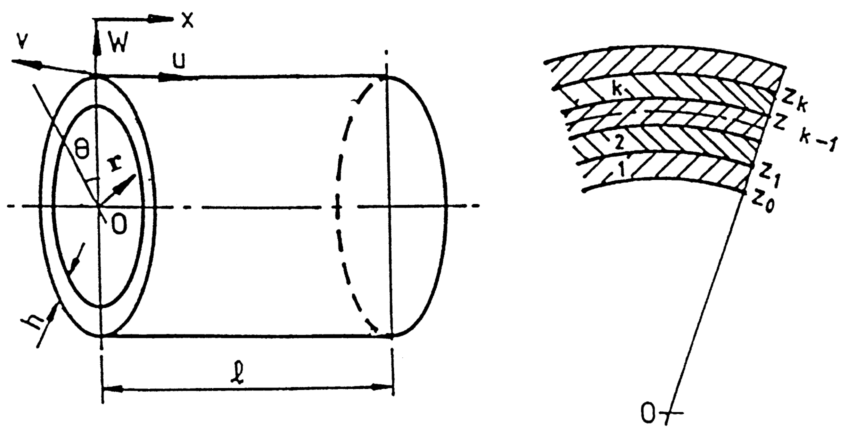

Fig. 1. Layered circular cylindrical shell of constant thickness: geometry.

dependence is given by

$$\sum_k (z_k^2 - z_{k-1}^2)\rho_k = 0, \tag{1}$$

where ρ_k is the density of the kth layer and z_k is the distance of the outer boundary of the kth layer from the reference surface.

The stress and moment resultants are expressed in terms of the longitudinal, circumferential and transverse displacement components u, v and w of a general point of the reference surface. The displacement components u, v, w are assumed in the form

$$\begin{aligned} u(x,\theta,t) &= U(x)\cos n\theta e^{i\omega t}, \\ v(x,\theta,t) &= V(x)\sin n\theta e^{i\omega t}, \\ w(x,\theta,t) &= W(x)\cos n\theta e^{i\omega t}, \end{aligned} \tag{2}$$

where x and θ are the longitudinal and rotational co-ordinates, ω is the angular frequency of vibration, t is the time and n is the circumferential node number. When $n = 0$, the vibration becomes axisymmetric.

Using Eq. (2) in the constitutive equations and the resulting expressions for the stress and moment resultants in the equations of equilibrium, the governing equations of motion in U, V and W are obtained in the form

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \{0\}, \tag{3}$$

where L_{ij} are linear differential operators in x.

The following non-dimensional parameters are introduced:

$$\begin{aligned} &\lambda = \ell\omega\sqrt{\frac{R_0}{A_{11}}}, && \text{a frequency parameter,} \\ &\delta_k = \frac{h_k}{h}, && \text{the relative layer thickness of the } k\text{th layer,} \\ &H = \frac{h}{r}, && \text{the thickness parameter,} \\ &L = \frac{\ell}{r}, && \text{a length parameter,} \\ &X = \frac{x}{\ell}, && \text{a distance co-ordinate} \\ &R = \frac{r}{\ell}, && \text{a radius parameter.} \end{aligned} \tag{4}$$

Here ℓ is the length of the cylinder, r is its radius, h_k is the thickness of the kth layer, h is the total thickness of the shell, R_0 is the inertial coefficient and A_{11} is a standard extensional rigidity coefficient. When only two layers are considered we write $\delta = \delta_1$ and hence $\delta_2 = 1 - \delta$. Clearly $0 \leqslant X \leqslant 1$.

The operators appearing in Eq. (3) are

$$L_{11} = \frac{d^2}{dX^2} - S_{10}\frac{n^2}{R^2} + \lambda^2, \tag{5}$$

$$L_{12} = \left(S_2 + S_{10} + \frac{S_5 + 2S_{11}}{R}\right)\frac{n}{R}\frac{d}{dX}, \tag{6}$$

$$L_{13} = -S_4\frac{d^3}{dX^3} + \left((S_5 + 2S_{11})\frac{n^2}{R^2} + \frac{S_2}{R}\right)\frac{d}{dX}, \tag{7}$$

$$L_{21} = -\left(S_2 + S_{10} + \frac{S_5 + S_{11}}{R}\right)\frac{n}{R}\frac{d}{dX}, \tag{8}$$

$$L_{22} = \left(S_{10} + \frac{3S_{11}}{R} + \frac{2S_{12}}{R^2}\right)\frac{d^2}{dX^2} - \left(S_3 + \frac{2S_6}{R} + \frac{S_5}{R^2}\right)\frac{n^2}{R^2} + \lambda^2, \tag{9}$$

$$L_{23} = \left(S_5 + 2S_{11} + \frac{S_8 + 2S_{12}}{R}\right)\frac{n}{R}\frac{d^2}{dX^2} - \left(S_6 + \frac{2S_9}{R}\right)\frac{n^3}{R^3} - \left(S_3 + \frac{S_6}{R}\right)\frac{n}{R^2}, \tag{10}$$

$$L_{31} = S_4\frac{d}{dX^3} - \left((S_5 + 2S_{11})\frac{n^2}{R^2} + \frac{S_2}{R}\right)\frac{d}{dX}, \tag{11}$$

$$L_{32} = \left(S_5 + 2S_{11} + \frac{S_8 + 4S_{12}}{R}\right)\frac{n}{R}\frac{d^2}{dX^2} - \left(S_6 + \frac{S_9}{R}\right)\frac{n^3}{R^3} - \left(S_3 + \frac{S_6}{R}\right)\frac{n}{R^2}, \tag{12}$$

$$L_{33} = -S_7\frac{d^4}{dX^4} + 2(S_8 + 2S_{12})\frac{n^2}{R^2}\frac{d^2}{dX^2} + \frac{2S_5}{R}\frac{d^4}{dX^2} - \left(S_9\frac{n^4}{R^4} + 2S_6\frac{n^2}{R^3} + S_3\frac{1}{R^2}\right) + \lambda^2, \tag{13}$$

where S_i $(i = 2, \ldots, 12)$ are as defined in Appendix A.

3. Method of solution

The differential equations in Eq. (3) contain derivatives of third order in U, second order in V and fourth order in W. Therefore, their form is not convenient to the solution procedure we propose to adopt. Hence, the equations are combined within themselves and a modified set of equations are derived.

To modify the equations, first of Eq. (3) is differentiated with respect to X once and used to eliminate $U'''(X)$ in the third equation. The modified set of equations are of order 2 in U, order 2 in V and order 4 in W, given by

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L^*_{31} & L^*_{32} & L^*_{33} \end{pmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \{0\}, \tag{14}$$

in which the new operators are

$$L^*_{31} = \left\{ S_4 S_{10} \frac{n^2}{R^2} - (S_5 + 2S_{11}) \frac{n^2}{R^2} - \frac{S_2}{R} \right\} \frac{d^2}{dx^2} + \lambda^2 S_4 \frac{d}{dx}, \tag{15}$$

$$\begin{aligned} L^*_{32} = {} & \left\{ \left(S_5 + 2S_{11} + \frac{S_8 + 4S_{12}}{R} \right) \frac{n}{R} - S_4 \left(S_2 + S_{10} + \frac{S_5 + 2S_{11}}{R} \right) \frac{n}{R} \right\} \frac{d^2}{dX^2} \\ & - \left(S_6 + \frac{S_9}{R} \right) \frac{n^3}{R^3} - \left(S_3 + \frac{S_6}{R} \right) \frac{n}{R^2}, \end{aligned} \tag{16}$$

$$\begin{aligned} L^*_{33} = {} & \left(2(S_8 + 2S_{12}) \frac{n^2}{R^2} + \frac{2S_5}{R} - S_4 (S_5 + 2S_{11}) \frac{n^2}{R^2} - S_4 \frac{S_2}{R} \right) \frac{d^2}{dX^2} \\ & + (S_4^2 - S_7) \frac{d^4}{dX^4} - S_9 \frac{n^4}{R^4} - 2S_6 \frac{n^2}{R^3} - S_3 \frac{1}{R^2} - \lambda^2. \end{aligned} \tag{17}$$

The boundary conditions considered are

(1) (C–C): both the ends clamped,
(2) (H–H): both the ends hinged.

Each pair of the boundary conditions imposed gives eight equations involving U, V and W. Thus, Eq. (14) together with each of these cases of boundary conditions constitute a well-defined two-point boundary value problem.

Numerical method of solution is resorted to, since no closed-form solution exists for these problems in their general form. The spline technique is used since it is relatively simple and elegant and uses a series of lower order approximation rather than a global higher order approximation, affording fast convergence and high accuracy.

The displacement functions $U(X)$, $V(X)$ and $W(X)$ are approximated by the cubic and quintic spline functions $U^*(X)$, $V^*(X)$ and $W^*(X)$, as stated below:

$$U^*(X) = \sum_{i=0}^{2} a_i X^i + \sum_{j=0}^{N-1} b_j (X - X_j)^3 \mathrm{H}(X - X_j),$$

$$V^*(X) = \sum_{i=0}^{2} c_i X^i + \sum_{j=0}^{N-1} d_j (X - X_j)^3 \mathrm{H}(X - X_j), \tag{18}$$

$$W^*(X) = \sum_{i=0}^{4} e_i X^i + \sum_{j=0}^{N-1} f_j (X - X_j)^5 \mathrm{H}(X - X_j).$$

Here N is the number of intervals into which the range [0,1] of X is divided and $H(X)$ is the Heaviside step function. The points of division $X = X_s = s/N$, $(s = 0, 1, \ldots, N)$ are chosen as the knots of the splines, as well as the collocation points. Imposing the condition that the differential equations given by Eq. (14) are satisfied by these splines at these points, a set of $3n + 3$ homogeneous equations in $3n + 11$ unknown spline coefficients: $a_i, b_j, c_i, d_j, e_k, f_j$ $(i = 0, 1, 2;\ k = 0, 1, 2, 3, 4;\ j = 0, 1, 2, \ldots, N - 1)$, are obtained. They are the field equations given in Appendix B.

Each of the sets (C–C) and (H–H) of the boundary conditions, on using the spline approximations, gives eight equations on the spline coefficients. These equations along with the field equations, reduce to a system which can be written in the form

$$[P]\{q\} = \Lambda[Q]\{q\}, \tag{19}$$

where $[P]$ and $[Q]$ are square matrices, $\{q\}$ is a column matrix and $\Lambda = 1/\lambda^2$ where λ is the frequency parameter. This is a generalized eigenvalue problem in which Λ or λ^2 is the eigenparameter and $\{q\}$ is the eigenvector.

4. Convergence and comparative studies

Since matrices of large orders are involved, numerical computations were done using double precision arithmetic. After a number of trials it was found that the number of knots of the spline function N could be taken as 12. The convergence studies made are not presented for want of space. As mentioned earlier only two layered shells were considered. Though high strength graphite (HSG), S-glass epoxy (SGE), PRD and steel (St) were considered, the results on HSG–SGE and HSG–PRD combinations only are presented graphically.

For the purpose of gaining conviction on the correctness of the analysis and accuracy of the results, the values of the natural frequencies ω obtained by Goncalves and Ramos [24], Arnold and Warburton [25] (experimental), Smith and Haft [26] and Au-Yang [27] (approximate), for a homogeneous shell under clamped – clamped conditions for the axial mode values $m = 1-3$ and the circumferential mode values $n = 2-6$ are compared with the corresponding results obtained by the present method. The parametric values used are $\ell = 511.2\,\mathrm{mm}$, $r = 216.2\,\mathrm{mm}$, $h = 1.5\,\mathrm{mm}$, $E = 1.83 \times 10^{11}\,\mathrm{N/m^2}$, $\nu = 0.3$, $\rho = 7492\,\mathrm{kg/m^3}$ [28]. The comparison is given in Table 1. The agreement of the current results, particularly with the experimental results, is very good. The

Table 1
Comparison of natural frequencies for cylinder shell clamped edges

Mode		Natural frequencies in Hz				
Axial mode m	Circumferential node n	Present method	Experimental results: Arnold and Warburton [25]	Analytical results: Smith and Haft [26]	Approximate method: Au-Yang [27]	Goncalves and Ramos [24]
1	2	1299	1240	1429	1485	1405
1	3	2240	2150	2335	2364	2221
1	4	3872	3970	4142	4147	4004
1	5	6262	6320	6500	6504	6360
1	6	9125	9230	9400	9396	9251
2	2	2660	2440	2682	2852	2904
2	3	2574	2560	2771	2813	2715
2	4	4043	4160	4335	4350	4212
2	5	6491	6475	6644	6655	6510
2	6	9276	9380	9840	9533	9384
3	2	4364	—[a]	4314	4579	4817
3	3	3483	3380	3570	3663	3623
3	4	4417	4540	4717	4749	4629
3	5	6728	6720	6903	6923	6788
3	6	9540	9540	9630	9763	9622

[a] No experimental results available from literature.

slightly higher differences with the other results quoted must be due to the differences in the formulations, methods of solution and numerical accuracy involved.

5. Results and discussion

Fig. 2 describes the manner of variation of the frequency parameter λ with respect to δ, the ratio of the thickness of the inner layer to the entire thickness. Figs. 2(a) and (b) correspond to the shell of HSG–SGE layer combination, with C–C and H–H boundary conditions respectively. The layer materials in respect of Figs. 2(c) and (d) are HSG–PRD, with the boundary conditions C–C and H–H, respectively. The first three meridional modes ($m = 1, 2, 3$) are considered here and in all the studies that follow. The continuous lines pertain to the inclusion of the coupling effect between extensional and flexural displacements ($B_{ij} \neq 0$) in the analysis, while the dashed lines correspond to ignoring this coupling effect ($B_{ij} = 0$). The values of the ratio of the shell thickness to radius (H) and the ratio of the shell length to the radius (L) are taken as 0.02 and 1.5, respectively.

It is clearly seen that as δ increases λ decreases in the case of HSG–SGE layers; and increases in the case of HSG–PRD layers. For the extreme values of δ, equal to 0 or 1, the shell becomes homogeneous, with the material of either of the two layers. It is seen that it is possible to attain a desired frequency, between these two extreme values (some times even outside this range of frequencies) by suitably choosing the value of δ. This must be interesting from the design point of

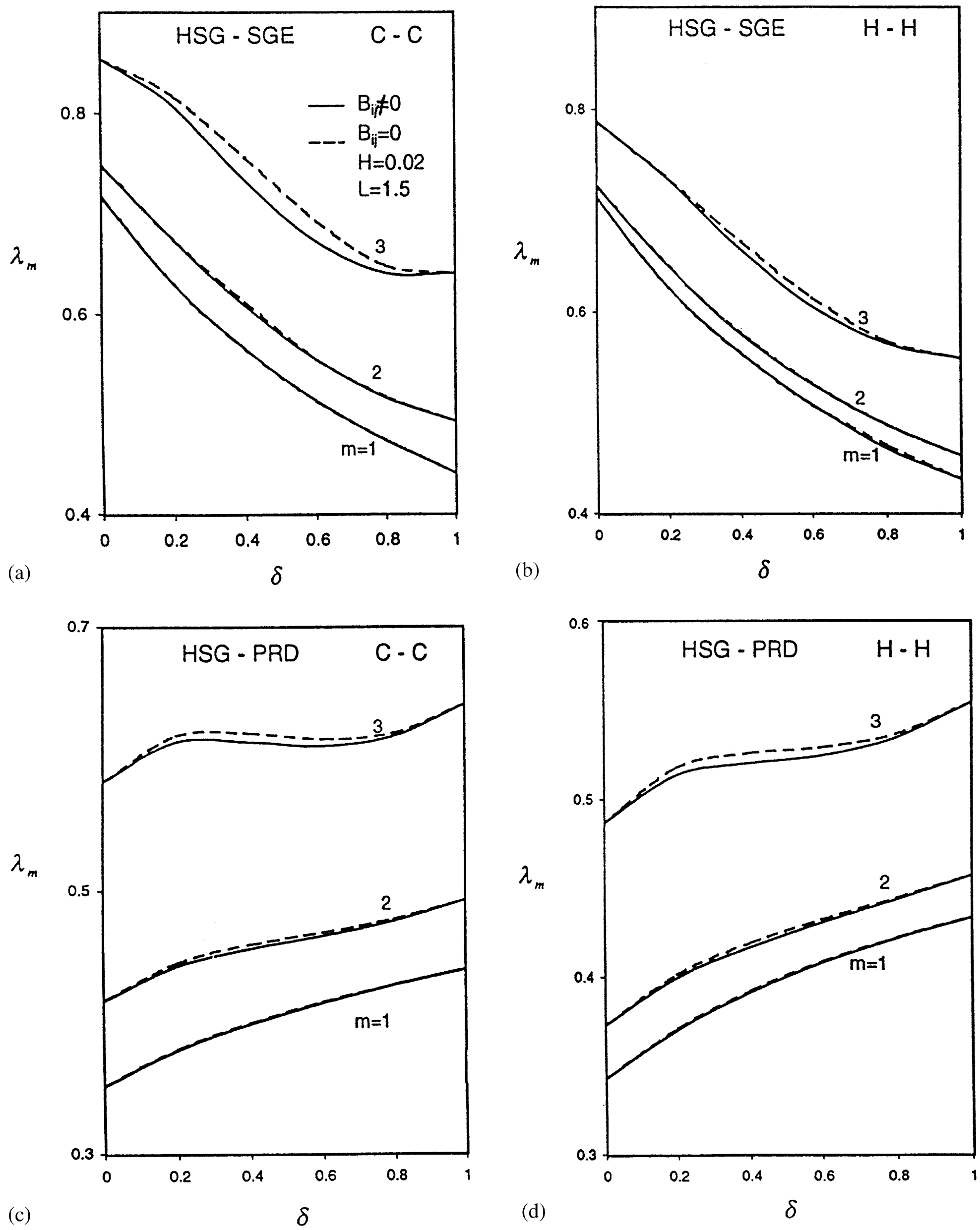

Fig. 2. Variation of frequency parameter with relative layer thickness and the effect of coupling and boundary conditions.

view. Under the same boundary conditions, λ_m (for each m considered) is highest for the homogeneous SGE shells (at $\delta = 0$ in Figs. 2(a) or (b)) and the least for homogeneous PRD shells (at $\delta = 0$ in Figs. 2(c) or (d)) and assumes a value in between them for homogeneous HSG shells (at $\delta = 1$ in (a), (b), (c) or (d)).

The neglect of coupling is seen to generally increase the frequency parameter values. This increase is almost unnoticeable in the case of the fundamental frequencies and becomes more and more significant with increasing mode number. The highest per cent increases in the case of HSG–SGE shell considered for the three modes under C–C conditions are 0.077%, 0.748% and 3.2%. The corresponding figures for the shell under H–H conditions are 0.104%, 0.274% and 1.438%. In general, these increases are so small that the neglect of the coupling may not introduce appreciable errors in computation of frequencies; but may result in some computational advantage. In the current work, however, the coupling was not ignored. When the other conditions are identical, frequencies are higher for shells under C–C conditions than for shells under H–H conditions.

These observations are in agreement with those made in the cases of laminated plates and conical shells [21–23].

The frequency parameter λ is explicitly a function of the length ℓ of the cylinder. Hence, when studying the influence of the length of the cylinder on its vibrational behaviour, the actual frequency ω and not λ is considered. Since a specific length is then to be prescribed, the thickness of the shell h is taken to be 1 cm. Fig. 3 describes how the length parameter L affects ω (in 10^4 Hz) for HSG–SGE and HSG–PRD layered cylinders under C–C and H–H boundary conditions, with $H = 0.05$ and $\delta = 0.4$. As L increases, ω is observed to decrease in general. The decrease is fast for very short shells (for $0.5 < L < 0.75$ in this study), the rate of decrease increasing with higher modes. There is a rapid change in the rate of decrease of ω in the interval $0.75 < L < 0.85$ after which the decrease is very low. The fundamental frequencies are almost constant for $L > 0.85$. Similar phenomena were observed in the case of conical shells also [21]. The per cent changes in ω_m ($m = 1, 2, 3$) over the range of $0.5 < L < 2.0$ for HSG–SGE shell are 246.34%, 708.02% and 1092.41% for C–C conditions, and 90.05%, 463.03% and 923.56% for H-H conditions.

Fig. 4 describes how the overall thickness parameter H affects the frequency parameter λ for shells described in the figure. λ increases with H, almost linearly. The rate of increase increases with the value of the mode number. The fundamental frequencies are mostly unaffected. The per cent changes in λ_m ($m = 1, 2, 3$) over the range $0.01 < H < 0.06$ for HSG–SGE shell are 3.519%, 22.931%, 69.01% for C–C boundary conditions, and 0.72%, 10.61%, 43.57% for H–H conditions.

Some typical mode shapes of axisymmetric vibrations of a HSG–SGE shell are presented in Fig. 5. The transverse deflections are predominant. For each meridional mode number the W- and U- displacements are normalized with respect to maximum W. The existing perfect symmetry in material and geometric properties, including the boundary conditions, is exhibited in the perfectly symmetric or antisymmetric mode shapes. The U-values are much lower than the corresponding W-values. This phenomenon is more pronounced in the case of H–H boundary conditions (for example, for the fundamental modes, $U_{max}/W_{max} \simeq 0.03$ and 0.005 for the C–C and the H–H conditions, respectively).

Figs. 6–9 relate to studies on asymmetric vibrations of layered cylindrical shells. The influence of the circumferential node number n on the frequency parameter λ is studied in Fig. 6. HSG–SGE and HSG–PRD shells, each under C–C and H–H boundary conditions, are considered. The $\lambda - n$ curves corresponding to inclusion and omission of the coupling between the flexural and extensional displacements ($B_{ij} \neq 0$ and $B_{ij} = 0$) are presented for $m = 1, 2, 3$. It is seen at the outset that n affects λ significantly. The effect is higher for lower meridional mode values. In general, λ

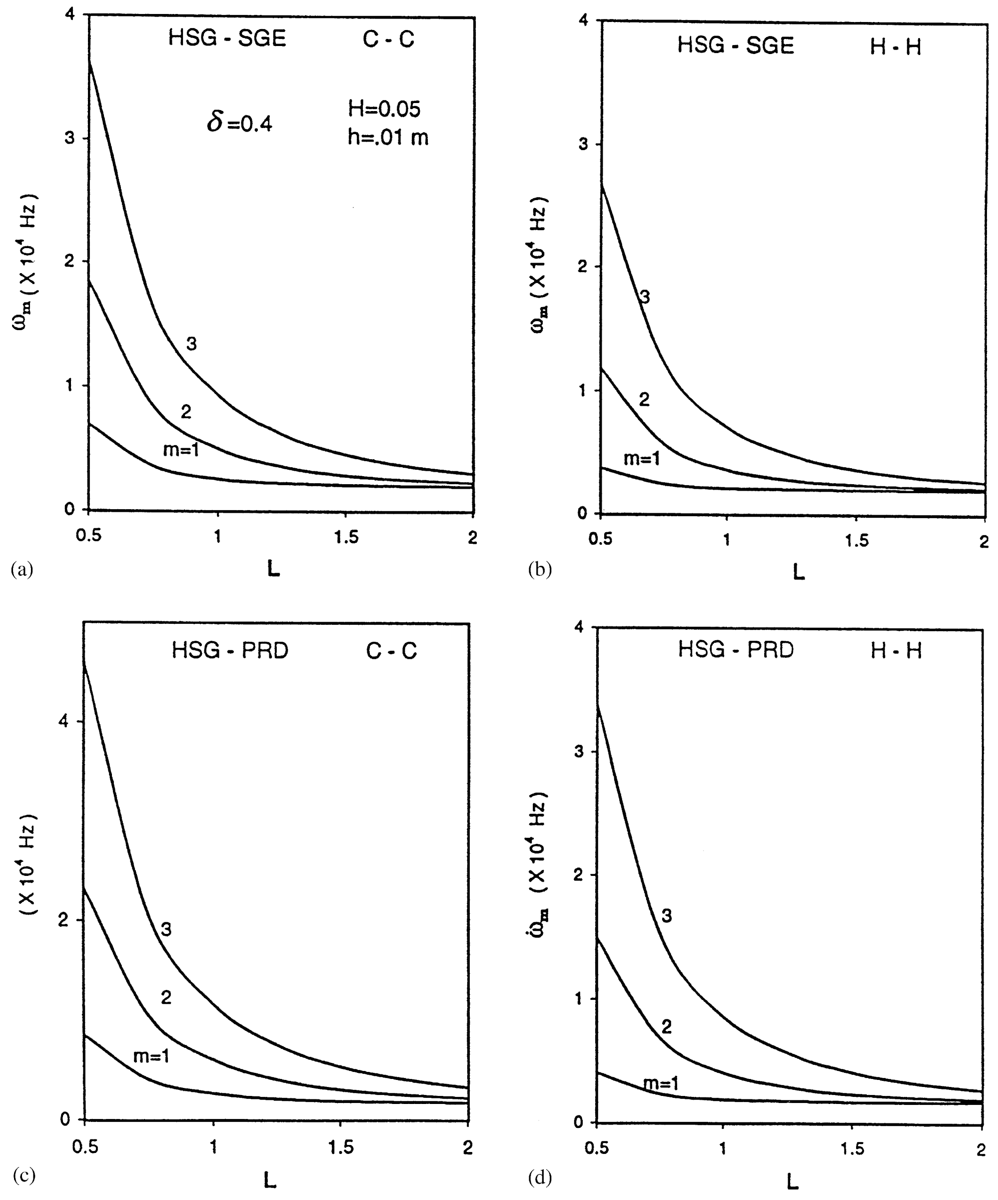

Fig. 3. Effect of length of the shell and boundary conditions on frequencies.

decreases with n up to some value of n and then increases. λ_{min} occurs approximately in the range $4<n<7$. The percentage of the maximum decrease in λ_1 over value of λ_1 for $n=0$ for the four cases considered in the figure are (a) 0.5%, (b) 0.104%, (c) 0.181% and (d) 0.191%. It can also be seen that the coupling affects the frequencies of asymmetric vibrations generally for all n considered. Omission of coupling results in increase in the values of frequencies. The effect is felt more for HSG–SGE layer combinations than for the other. It is the higher for higher value of m

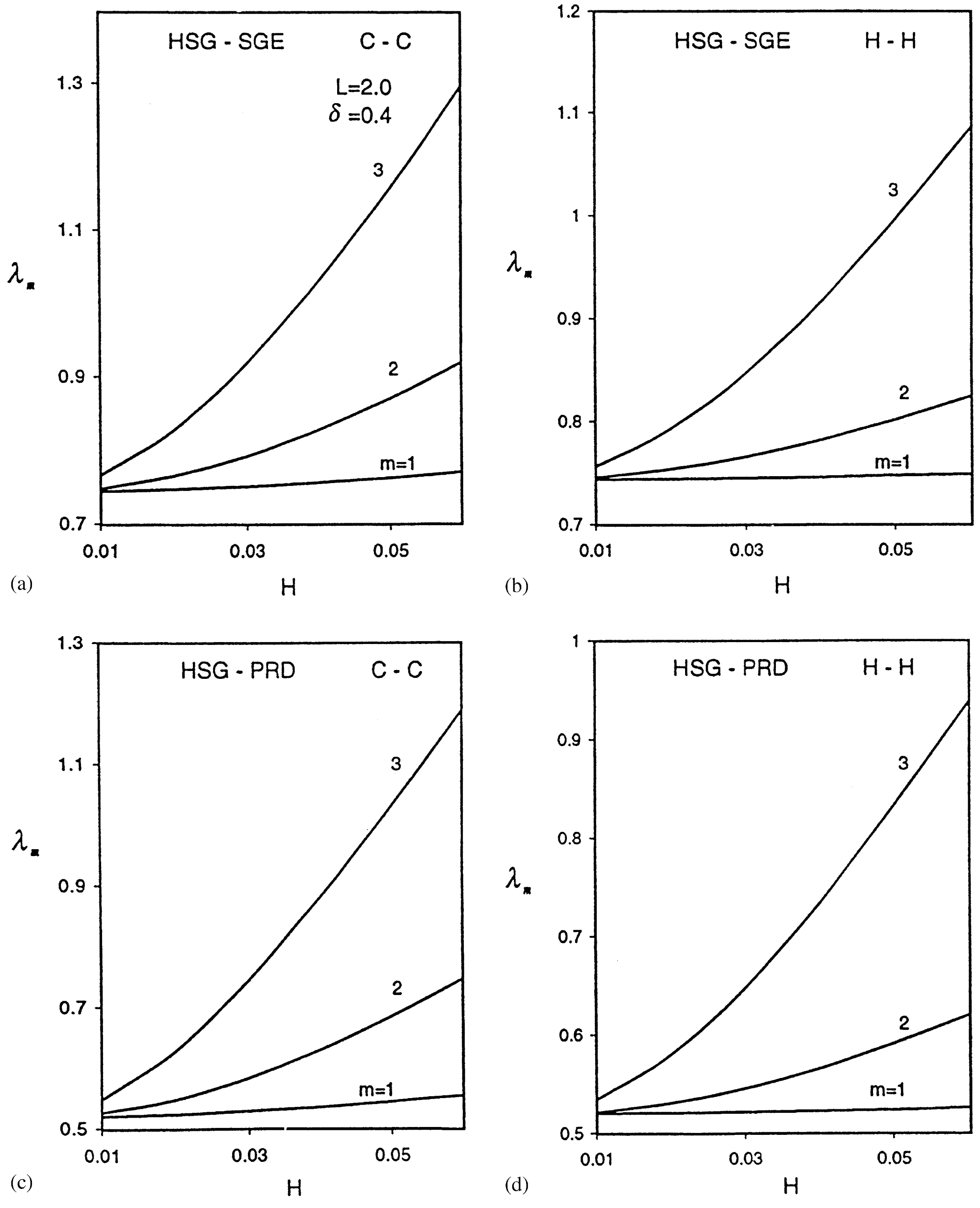

Fig. 4. Effect of thickness parameter and boundary conditions on frequency parameter.

for any fixed n. However, the relative increase is always so small that the neglect of this coupling will not introduce any appreciable error in the values of λ.

In Fig. 7 the influence of the length parameter L on the frequencies ω_m are analysed for two cases of asymmetric vibrations, $n = 4$ and 8, of a HSG–SGE shell under C–C and H–H conditions. The pattern of this influence is similar in nature to the corresponding cases of axisymmetric vibrations and differ only in magnitudes. The steep decrease in values of ω_m up to

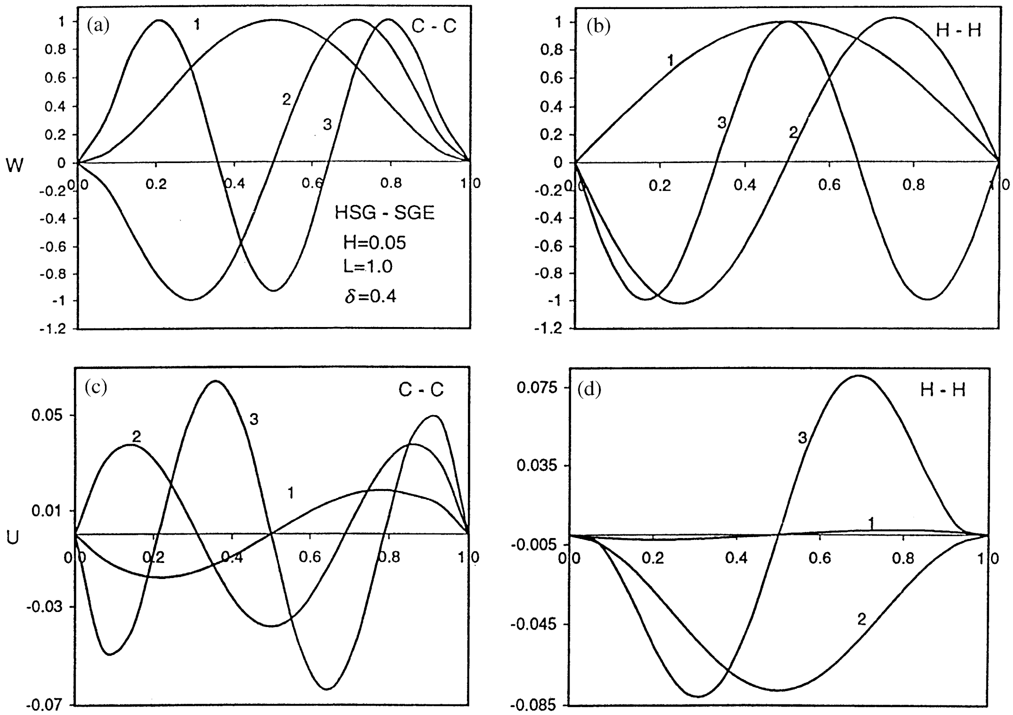

Fig. 5. Mode shapes of axisymmetric vibrations.

some values of L and slow decrease afterwards are the typical phenomena seen here as in the cases of axisymmetric and asymmetric vibrations of conical shells and circular plates also. The per cent changes in ω_1 over the range of L considered, for the four cases depicted are, respectively (a) 749.71%, (b) 411.15%, (c) 281.85% and (d) 133.47%.

The dependence of λ_m on H, the thickness parameter of the same shell under the same two types of boundary conditions for the same two cases of asymmetric vibrations is shown in Fig. 8. As in the case of $n = 0$, λ_m increases almost linearly with H. The linearity is more striking for $n = 8$ than for $n = 4$. The effect is more pronounced with higher values of m. The per cent change in λ_1 for the four cases considered are: (a) 78.89%, (b) 59.64%, (c) 418.87% and (d) 411.14%. Thus, the effect is higher for higher n under the same boundary conditions; and higher for C–C boundary conditions for a fixed n.

The mode shapes of asymmetric vibrations for $n = 4$ and 8 of a typical shell, of the description given in the figure, are presented in Fig. 9. The perfect symmetry or asymmetry in the mode shapes are as expected and noteworthy. Some of the mode shapes for $n = 4$ and 8 coalesce into one and the same. Indeed, the axial vibration modes are similar for all values of n considered. The axial vibration (W) modes obtained for $m = 1, 2, 3$ compare very well with the mode shapes presented by Goncalves and Ramos [24]. The transverse displacements are dominant in the considered asymmetric vibrations also. The less dominant are the rotational, and the least dominant are the extensional. For $m = 1$, the values of U_{max}/W_{max} and V_{max}/W_{max} are 0.07 and 0.41 for $n = 4$; and 0.074 and 0.117 for $n = 8$.

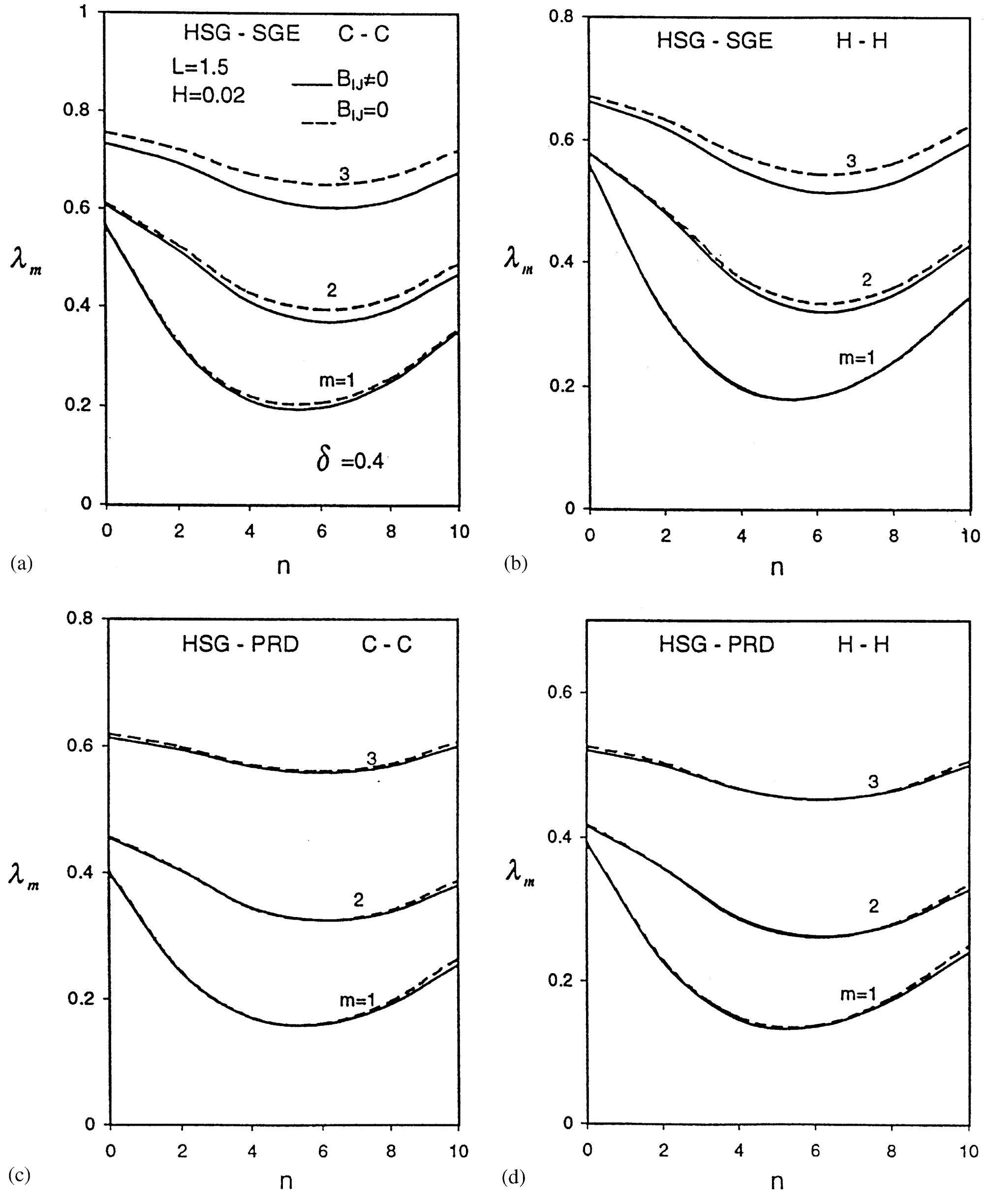

Fig. 6. Variation of frequency parameter with circumferential node number and the effect of coupling.

6. Conclusions

Layering of the shell wall influences the natural frequencies of the vibration of the cylindrical shell. The materials of the layers, as well as the relative thickness between them affect the frequencies. A desired frequency of vibration, within a range of frequencies, may be obtained by a proper choice of the relative thickness of layers among the chosen materials of the layers.

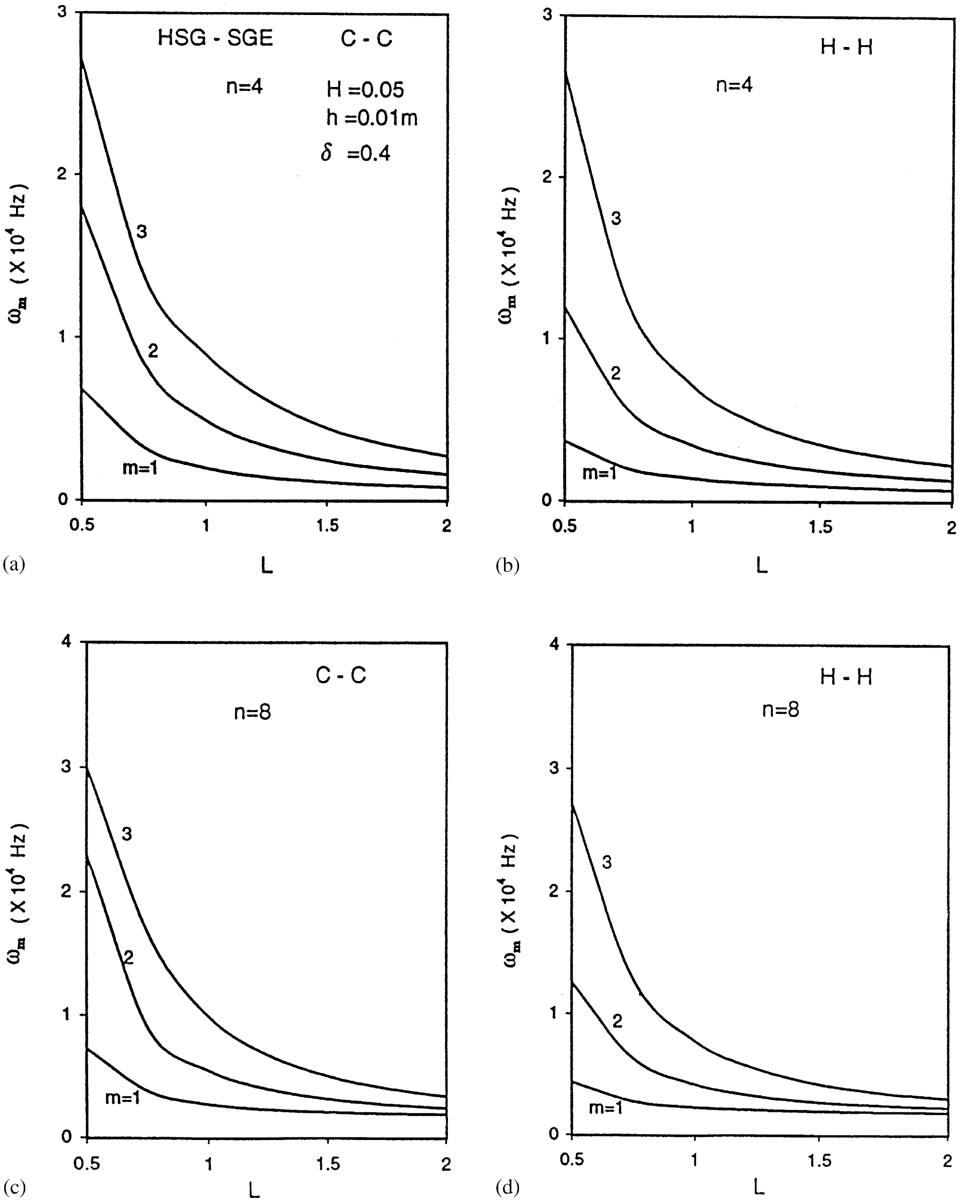

Fig. 7. Effect of length of the shell and boundary conditions on frequencies of asymmetric vibrations.

The clamped–clamped boundary conditions give rise to higher frequencies in comparison with the hinged–hinged boundary conditions. The neglect of coupling between the longitudinal and radial displacements results in increase of frequency values in general. However the increase is negligibly small.

With increase in length of the cylinder the natural frequencies of vibration decrease in general. The decrease is rapid for short shells and meager for long shells. With the increase in thickness of

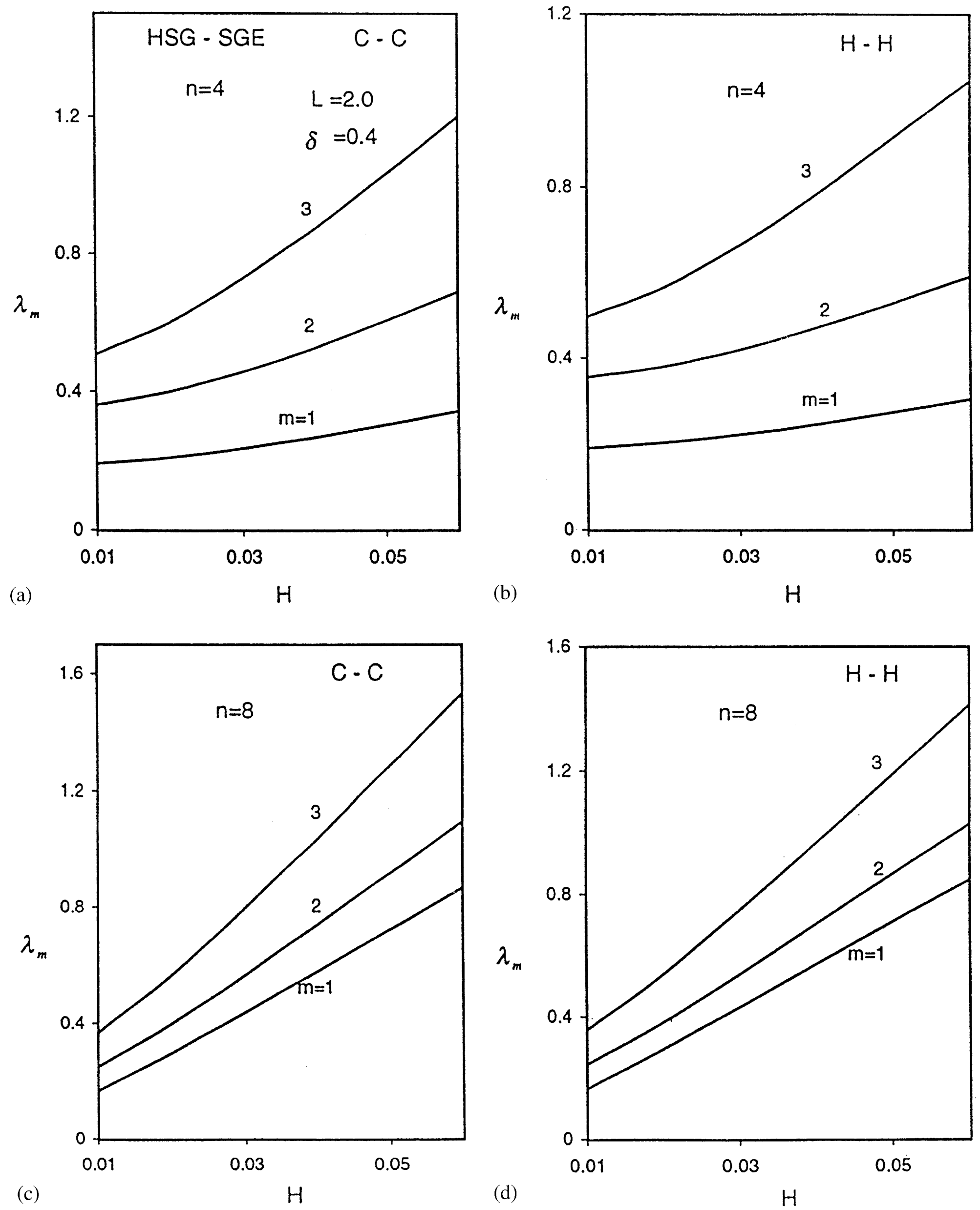

Fig. 8. Effect of thickness parameter and boundary conditions on frequency parameter of asymmetric vibrations.

shell wall the frequencies increase almost linearly. Both the above effects are felt more for higher longitudinal mode numbers.

All the above phenomena are observed in axisymmetric as well as asymmetric vibrations. With increase in circumferential node number the frequencies decrease initially and then increase. All the above characteristics except those described in the first paragraph are in close agreement with the vibrational behaviour of homogeneous thin cylindrical shells.

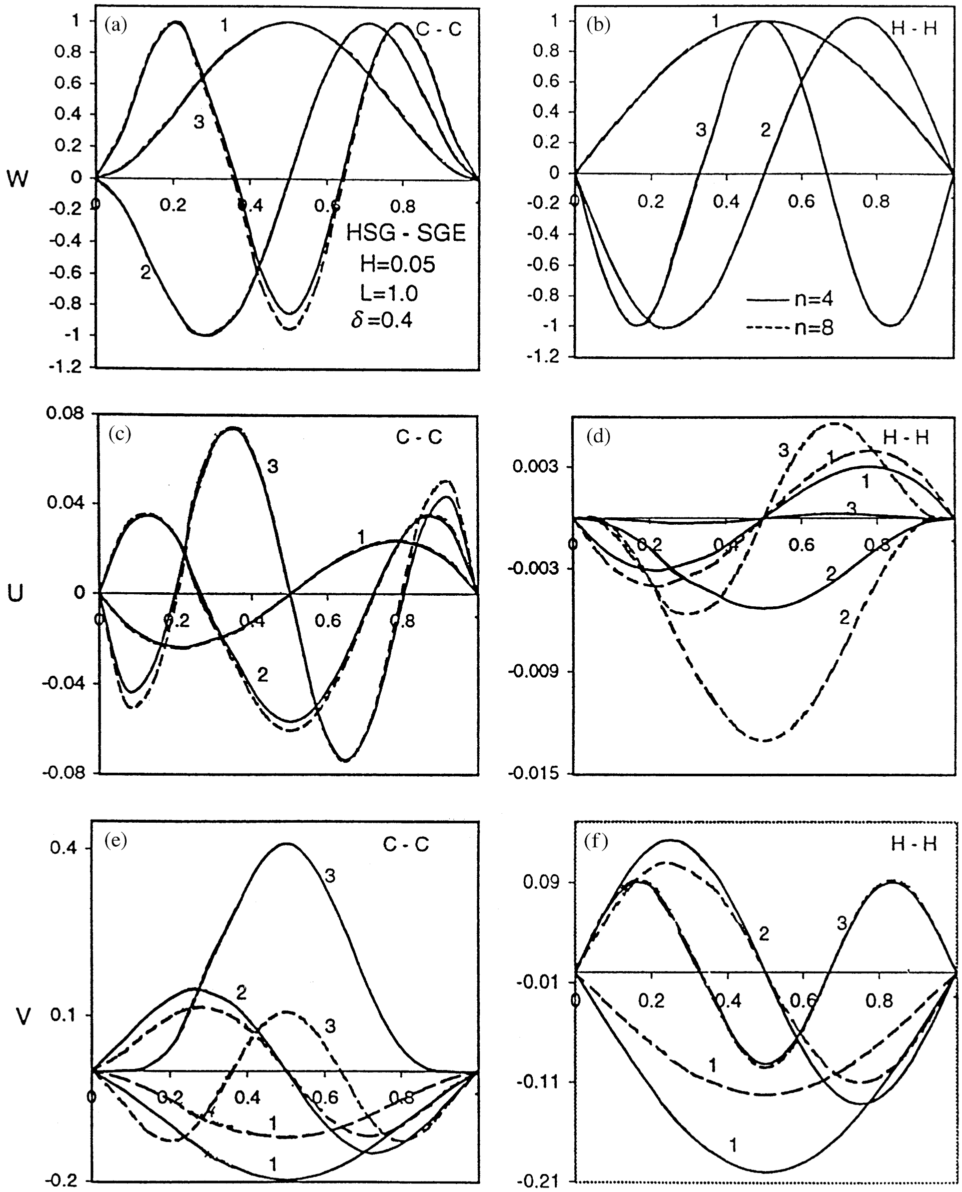

Fig. 9. Mode shapes of asymmetric vibraitons.

Appendix A

The quantities S_i ($i = 2, 3, \ldots, 12$) appearing in Eqs. (5)–(13) are defined by

$$S_2 = A_{12}/A_{11}, \quad S_3 = A_{22}/A_{11}, \quad S_4 = B_{11}/\ell A_{11},$$

$$S_5 = B_{12}/\ell A_{11}, \quad S_6 = B_{22}/\ell A_{11}, \quad S_7 = D_{11}/\ell^2 A_{11},$$

$$S_8 = D_{12}/\ell^2 A_{11}, \quad S_9 = D_{22}/\ell^2 A_{11}, \quad S_{10} = A_{66}/A_{11},$$

$$S_{11} = B_{66}/\ell A_{11}, \quad S_{12} = D_{66}/\ell^2 A_{11},$$

where

$$A_{ij} = \sum_k S_{ij}^{(k)}(z_k - z_{k-1}), \quad B_{ij} = \frac{1}{2}\sum_k S_{ij}^{(k)}(z_k^2 - z_{k-1}^2)$$

and

$$D_{ij} = \frac{1}{3}\sum_k S_{ij}^{(k)}(z_k^3 - z_{k-1}^3),$$

with

$$S_{11}^{(k)} = \frac{E_x^{(k)}}{1 - \upsilon_{x\theta}^{(k)}\upsilon_{\theta x}^{(k)'}} \quad S_{12}^{(k)} = \frac{\upsilon_{\theta x}^{(k)} E_x^{(k)}}{1 - \upsilon_{x\theta}^{(k)}\upsilon_{\theta x}^{(k)}},$$

$$S_{22}^{(k)} = \frac{E_\theta^{(k)}}{1 - \upsilon_{x\theta}^{(k)}\upsilon_{\theta x}^{(k)'}} \quad S_{66}^{(k)} = G_{x\theta}^{(k)}.$$

A_{ij}, B_{ij} and D_{ij} are, respectively, the extensional rigidities, the bending–stretching coupling rigidities and the bending rigidities.

Appendix B

The field equations appearing in the text are

$$\begin{aligned}
&A_0 a_0 + A_0 \frac{s}{N} a_1 + \left(2 + A_0 \frac{s^2}{N^2}\right) a_2 + \sum_{j=0}^{N-1}\left(6\frac{(s-j)}{N} + A_0 \frac{(s-j)^3}{N^3}\right) b_j \\
&+ A_1 C_1 + 2A_1 \frac{s}{N} C_2 + 3A_1 \sum_{j=0}^{N-1} \frac{(s-j)^2}{N^2} d_j + A_2 e_1 + 2A_2 \frac{s}{N} e_2 + \left(3A_2 \frac{s^2}{N^2} + 6A_3\right) e_3 \\
&+ \left(4A_2 \frac{s^3}{N^3} + 24A_3 \frac{s}{N}\right) e_4 + \sum_{j=0}^{N-1}\left(60A_3 \frac{(s-j)^2}{N^2} + 5A_2 \frac{(s-j)^4}{N^4}\right) f_j \\
&+ \lambda^2\left(a_0 + a_1 \frac{s}{N} + a_2 \frac{s^2}{N^2} + \sum_{j=0}^{N-1} b_j \frac{(s-j)^3}{N^3}\right) = 0,
\end{aligned} \tag{B.1}$$

$$B_0a_1 + 2B_0\frac{s}{N}a_2 + 3B_0\sum_{j=0}^{N-1}\frac{(s-j)^2}{N^2}b_j + B_2c_0 + B_2\frac{s}{N}c_1 + \left(2B_1 + B_2\frac{s^2}{N^2}\right)c_2$$
$$+ \sum_{j=0}^{N-1}\left(6B_1\frac{(s-j)}{N} + B_2\frac{(s-j)^3}{N^3}\right)d_j + B_4e_0 + B_4\frac{s}{N}e_1 + \left(2B_3 + B_4\frac{s^2}{N^2}\right)e_2$$
$$+ \left(6B_3\frac{s}{N} + B_4\frac{s^3}{N^3}\right)e_3 + \left(12B_3\frac{s^2}{N^2} + B_4\frac{s^4}{N^4}\right)e_4 + \sum_{j=0}^{N-1}\left(20B_3\frac{(s-j)^3}{N^3} + B_4\frac{(s-j)^5}{N^5}\right)f_j$$
$$+ \lambda^2\left(c_0 + c_1\frac{s}{N} + c_2\frac{s^2}{N^2} + \sum d_j\frac{(s-j)^3}{N^3}\right) = 0, \tag{B.2}$$

$$E_0a_1 + 2E_0\frac{s}{N}a_1 + 3E_0\sum_{j=0}^{N-1}\frac{(s-j)^2}{N^2}b_j + E_1c_1 + 2E_2\frac{s}{N}c_1 + \left(2E_2 + E_1\frac{s^2}{N^2}\right)c_2$$
$$+ \sum_{j=0}^{N-1}\left(6E_2\frac{(s-j)}{N} + E_1\frac{(s-j)^3}{N^3}\right)d_j + E_3e_0 + E_3\frac{s}{N}e_1 + \left(2E_4 + E_3\frac{s^2}{N^2}\right)e_2$$
$$+ \left(6E_4\frac{s}{N} + E_3\frac{s^3}{N^3}\right)e_3 + \left(12E_4\frac{s^2}{N^2} + 24E_5 + E_3\frac{s^4}{N^4}\right)e_4$$
$$+ \sum_{j=0}^{N-1}\left(20E_4\frac{(s-i)^3}{N^3} + 120E_5\frac{s-j}{N} + E_3\frac{(s-j)^5}{N^5}\right)f_j$$
$$+ \lambda^2\left(e_0 + e_1\frac{s}{N} + e_2\frac{s^2}{N^2} + e_3\frac{s^3}{N^3} + e_4\frac{s^4}{N^4} + \sum_{j=0}^{N-1}\frac{(s-j)^5}{N^5}f_j + A_3a_1\right.$$
$$\left. + 2A_3\frac{s}{N}a_2 + 3A_3\sum_{j=0}^{N-1}\frac{(s-j)^2}{N^2}b_j\right) = 0 \quad (s = 0, 1, 2, \ldots, N) \tag{B.3}$$

Here

$$A_0 = -S_{10}\frac{n^2}{R^2}, \quad A_1 = \left(S_2 + S_{10} + \frac{S_5 + 2S_{11}}{R}\right)\frac{n}{R},$$
$$A_2 = (S_5 + 2S_{11})\frac{n^2}{R^2} + \frac{S_2}{R}, \quad A_3 = -S_4,$$
$$B_0 = -\left(S_2 + S_{10} + \frac{S_5 + S_{11}}{R}\right)\frac{n}{R},$$
$$B_1 = S_{10} + \frac{3S_{11}}{R} + \frac{2S_{12}}{R},$$
$$B_2 = -\left(S_3 + \frac{2S_2}{R} + \frac{S_9}{R^2}\right)\frac{n^2}{R^2},$$

$$B_3 = \left(S_5 + 2S_{11} + \frac{S_8 + 2S_{12}}{R}\right)\frac{n}{R},$$

$$B_4 = -\left(S_6 + \frac{S_9}{R}\right)\frac{n^3}{R^3} - \left(S_3 + \frac{S_6}{R}\right)\frac{n}{R^2},$$

$$E_0 = S_4 S_{10}\frac{n^2}{R^2} - (S_5 + 2S_{11})\frac{n^2}{R^2} - \frac{S_2}{R},$$

$$E_1 = -\left(S_6 + \frac{S_9}{R}\right)\frac{n^3}{R^3} - \left(S_3 + \frac{S_6}{R}\right)\frac{n}{R^2},$$

$$E_2 = \left(S_5 + 2S_{11} + \frac{S_8 + 4S_{12}}{R}\right)\frac{n}{R} - S_4\left(S_2 + S_{10} + \frac{S_5 + 2S_{11}}{R}\right)\frac{n}{R},$$

$$E_3 = -S_9\frac{n^4}{R^4} - S_3\frac{1}{R^2} - 2S_6\frac{n^2}{R^3},$$

$$E_4 = 2(S_8 + 2S_{12})\frac{n^2}{R^2} + \frac{2S_5}{R} - S_4(S_5 + 2S_{11})\frac{n^2}{R^2} - \frac{S_4 S_2}{R},$$

$$E_5 = S_4^2 - S_7$$

and

$$\sum_{j=0}^{s-1}\{\ldots\} = 0 \quad \text{for } s = 0. \tag{B.4}$$

References

[1] A.W. Leissa, Vibration of Shells, NASA SP-288, 1973.
[2] E.H. Baker, G. Herrmann, Vibrations of orthotropic cylindrical sandwich shells under initial stress, American Institute of Aeronautics Journal 29 (1966) 963–975.
[3] S. Markus, Refined theories of damped axisymmetric vibrations of double-layered cylindrical shells, Journal of Mechanical Engineering Science 21 (1) (1979) 33–37.
[4] S. Chonan, Acoustic characteristics and the design of two layered cylindrical shells, JSME International Journal 31 (3) (1988) 561–567.
[5] S.B. Dong, Free vibation of laminated orthotropic cylindrical shells, Journal of Acoustical Society of America 44 (1968) 1628–1635.
[6] R.M. Jones, Buckling and vibration of cross-ply laminated circular cylindrical shells, American Institute of Aeronautics Journal 13 (1975) 664–671.
[7] L.H. Donnell, Stability of thin-walled tubes under torsion, NASA Report 479, Washington, DC, 1933.
[8] C.W. Bert, J.L. Baker, D.M. Egle, Free vibration of multilayered anisotropic cylindrical shells, Journal of Composite Materials 3 (1969) 480–499.
[9] Y. Stavsky, R. Loewy, Vibration analysis of composite annular plates by an integral equation technique, Journal of Sound and Vibration 95 (2) (1971) 143–150.
[10] J.B. Greenberg, Y. Stavsky, Buckling and vibration of orthotropic composite cylindrical shells, Acta Mechanica 36 (1980) 15–29.

[11] A.E.H. Love, A Treatise on the Mathematical Theory of Elasticity, Fourth Edition, Cambridge University Press, Cambridge, 1952.
[12] J.N. Reddy, Finite-element modelling of layered anisotropic composite plates and shells–a review of recent research, Shock Vibration Digest 13 (1981) 3–12.
[13] Y. Narita, Y. Ohta, G. Yamada, Y. Kobayashi, Analytical method for vibration of angle-ply cylindrical shells having arbitrary edgers, American Institute of Aeronautics Journal 30 (1992) 790–796.
[14] A.A. Khdeir, J.N. Reddy, Influence of edge conditions on the model characteristics of cross-ply laminated shells, Computers and Structures 34 (6) (1990) 817–826.
[15] T. Mizusawa, H. Kito, Vibration of cross-ply laminated cylindrical panels by the spline strip method, Computers and Structures 57 (2) (1995) 253–265.
[16] K.V. Lam, C.T. Loy, Effects of boundary conditions on frequencies of a multilayered cylindrical shell, Journal of Sound and Vibration 188 (3) (1995) 363–384.
[17] W.G. Bickley, Piecewise cubic interpolation and two point boundary problems, Computer Journal 11 (1968) 206–208.
[18] S.R. Soni, K. Sankara Rao, Vibration of non-uniform rectangular plates: a spline technique method of solution, Journal of Sound and Vibration 35 (1974) 35–45.
[19] T. Irie, G. Yamada, R. Kanda, Free vibration of rotating non-uniform discs: spline interpolation technique calculation, Journal of Sound and Vibration 66 (1979) 13–23.
[20] T. Irie, G. Yamada, Analysis of free vibration of annular plate of variable thickness by use of a spline technique method, JSME Bulletin 23 (1980) 286–292.
[21] P.V. Navaneethakrishnan, K. Chandrasekaran, A spline function analysis of axisymmetric free vibrations of layered conical shells and plates of variable thickness, American Society of Mechanical Engineers Structural Vibration and Acoustics 18 (1989) 153–159.
[22] P.V. Navaneethakrishnan, K. Chandrasekaran, N. Ravi Srinivas, Axisymmetric vibration of layered tapered plates, American Society of Mechanical Engineers Journal of Applied Mechanics 59 (1992) 1041–1043.
[23] P.V. Navaneethakrishnan, Vibration of layered shells and plates: a unified formulation and spline function study, Proceedings of the International Noise and Vibration Control Conference NOISE-93, Vol. 7, St. Petersburg, Russia, 1993, pp. 71–76.
[24] P.B. Goncalves, N.R.S.S. Ramos, Numerical method for vibration analysis of cylindrical shells, American Society of Civil Engineers Journal of Engineering Mechanics 123 (1997) 544–550.
[25] R.N. Arnold, G.B. Warburton, Flexural vibrations of the walls of thin cylindrical shells having freely supported ends, Proceeding of the Royal Society of London 197 (1949) 238–256.
[26] B.L. Smith, E.E. Haft, Natural frequencies of clamped cylindrical shells, Journal of Aeronautics and Astronautics 6 (1968) 720–721.
[27] M.K. Au-Yang, Natural frequency of cylindrical shells and panels in vacuum and in a fluid, Journal of Sound and Vibration 57 (1978) 341–355.
[28] H. Chung, Free vibration analysis of circular cylindrical shells, Journal of Sound and Vibration 74 (1981) 331–350.